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The damped nonlinear wave equation $W_{tt} = \sigma(W)_{xx} - \gamma W_t$ is formally equivalent to a quasilinear system of the form

$$\begin{cases} W_t - V_x = 0 \\ V_t - \sigma(W)_x + \gamma V = 0 \end{cases}$$

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which arises, in particular, in models of shearing flow in a nonlinear viscoelastic fluid. In such models, it has been shown that when the system is strictly hyperbolic, i.e., $\sigma'(0)>0$ and the initial data is small (so that $\sigma'(W)>0$ for as long as solutions exist) smooth solutions will fail to exist globally in time if the gradients of the initial data are too large. In this paper, we show that similar results hold if $\sigma'(0)=0$ or if $\sigma'(\gamma)<0$ for $|\gamma|$ sufficiently large; we also derive growth estimates for the $L_2$ norm of a maximally-defined smooth solution.

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# On the Damped Nonlinear Evolution Equation

$$w_{tt} = \sigma(w)_{xx} - \gamma w_t$$

by


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## 1. Introduction

The first order quasilinear system

$$\text{(S)} \qquad \begin{cases} w_t - v_x = 0 \\ v_t - \sigma(w)_x + \gamma v = 0 \end{cases} \qquad (\gamma \geq 0)$$

arises in a variety of ways in several areas of applied mathematics; the problem of establishing global existence and nonexistence theorems for initial-boundary value problems associated with such systems has been the subject of much investigation during the past fifteen years. If (in a simple connected domain) in $(x,t)$ space we set $v = y_t$, $w = y_x$ then (S) is transformed into the dissipative (if $\gamma > 0$) quasi-linear wave equation

$$(1.1) \qquad y_{tt} + \gamma y_t = \sigma(y_x)_x \equiv \lambda(y_x) y_{xx}$$

With $\gamma = 0$, this equation was studied by Zabusky [1] under the assumption that $\lambda^2(\zeta) = (1 + \epsilon\zeta)$ and that the initial and boundary data are of the form

$$(1.2) \qquad \begin{cases} y(0,t) = y(L,t), \; t > 0 \\ y(x,0) = \tilde{y}_0(x), \; y_t(x,0) = 0, \; 0 \leq x \leq L \end{cases}.$$

The initial-boundary value problem (1.1), (1.2) (with $\gamma = 0$) serves to model the transverse vibrations of a finite nonlinear string. By employing the method of Riemann invariants Zabusky proved that a smooth solution of (1.1), (1.2) must break down in finite time as a result of some second derivative of $y(x,t)$ becoming infinite; this, in turn implies the development of shocks in the solution $(v,w)$ of the quasilinear system (S). Using a different argument (but one also based on Riemann invariants) Lax [2] in

1964 extended Zabusky's finite-time breakdown results for (1.1), (1.2) to the case of a general positive function $\lambda(\zeta)$ satisfying $|\lambda'(\zeta)| \geq \lambda_o > 0$ ; the assumption of positive $\lambda$ is equivalent to assuming that the system (S) , with $\gamma \geq 0$ , is strictly hyperbolic, i.e. that $\sigma'(\zeta) > 0$ , $\forall \zeta \in R^1$ . In [3] MacCamy and Mizel again studied (1.1), with $\gamma = 0$ , subject to initial and boundary data of the form

$$(1.3) \qquad \begin{array}{ll} y(0,t) = 0 \ , \ y(L,t) = 0 \ , & t > 0 \\ y(x,0) = 0 \ , \ y_t(x,0) = \tilde{y}_1(x) \ , & 0 \leq x \leq L \end{array}$$

and extended the breakdown results of Zabusky and Lax to general functions $\lambda(\zeta)$ satisfying $\lambda(\zeta) > 0$ , $\forall \zeta \in R^1$ , $\lambda(0) = 1$ , and $\lambda'(\zeta) \lessgtr 0$ for $\zeta \gtrless 0$ ; they also proved that if either of the integrals

$$\int_o^{\infty} \lambda(\zeta)d\zeta \ \ \underline{\text{or}} \ \ \int_o^{-\infty} \lambda(\zeta)d\zeta$$

is finite, then there exist intervals on the x-axis in which the solution must exist for all time even though it must breakdown for some x-values outside these intervals. The latter results of MacCamy and Mizel can be extended to more general initial conditions of the type

$$y(x,0) = \tilde{y}_o(x) \ , \ y_t(x,0) = \tilde{y}_1(x), \ 0 \leq x \leq L$$

with $\tilde{y}_o^2(x) + \tilde{y}_1^2(x) \not\equiv 0$ but no pair of initial data of the form $(\tilde{y}_o(x),0)$ can be found which allows for global existence in time of a smooth solution in some $x$ interval. In a later series of papers [4], [5] MacCamy, Mizel, and Greenberg considered the damped nonlinear wave equation

$$(1.3) \qquad y_{tt} = \frac{\partial}{\partial x}(\sigma(y_x) + \rho(y_x)y_{xt})$$

and proved that initial-boundary value problems associated with (1.3) always have smooth global solutions which are, in fact, asymptotically stable, no matter how large the initial data $\tilde{y}_0(x)$ , $\tilde{y}_1(x)$ are.

Much of the more interesting work concerning the damped versions $(\gamma > 0)$ of (1.1), and the related system (S) , is of a more recent vintage; there has also been a concerted effort on the part of various researchers to resolve the problem of global existence vs. nonexistence of smooth solutions to nonlinear one-dimensional integrodifferential equations which arise in several theories of nonlinear viscoelastic response and which involve damping mechanisms that are sometimes formally equivalent to that present in the system (S) but which are often more subtle.

In a significant piece of work Nishida [6] has recently considered the initial-value problem for the damped quasilinear system of equations (S) and has proven, using a Riemann invariant argument, that unlike the situation in the undamped case $(\gamma = 0)$ , global smooth solutions do exist if the initial data are small in an appropriate sense; to be precise, Nishida considers $\sigma(\zeta)$ such that $\sigma'(\zeta) > 0$ for $|\zeta| < \delta$ , with $\sigma(\cdot) \in C^2(|\zeta| < \delta)$ , defines the Riemann invariants $r$ , $s$ via

$$r = \phi(w) - v \quad , \quad s = - \phi(w) - v$$

where

$$\phi(w) = \int_0^w \sqrt{\sigma'(\zeta)}\, d\zeta$$

and assumes, in his proof of global existence, that $r(x,0)$ , $s(x,0)$ , as determined by the initial data $v(x,0)$ , $w(x,0)$ associated with (S) ,

satisfy

$$r(x,0)\ ,\ s(x,0) \in C^1(R^1) \quad \text{with}$$

$$\sup|r(x,0)| + \sup|s(x,0)| < \min\{2\phi(\delta)\ ,\ -2\phi(-\delta)\} \tag{1.4}$$

$$\sup\left|\frac{dr(x,0)}{dx}\right| < +\infty\ ,\ \sup\left|\frac{ds(x,0)}{dx}\right| < +\infty \tag{1.5}$$

and, for $\epsilon > 0$ sufficiently small,

$$(\sup|r(x,0)| + \sup|s(x,0)|) + (\sup|\frac{dr}{dx}(x,0)| + \sup|\frac{ds}{dx}(x,0)|) < \epsilon \tag{1.6}$$

Thus, under Nishida's hypotheses the system (S) is strictly hyperbolic in $\Omega = \{(v,w)|v \in R^1 . |w| < \delta\}$ ; this corresponds to the assumption that the damped quasilinear wave equation (1.1) is hyperbolic in a neighborhood of $y_x = 0$ . Nishida also obtains global existence and decay to zero, in the $L^\infty$ norm, as $t \to +\infty$ , of a unique smooth solution of (1.1) by adopting Matsumura's modification [7] of an $L^2$-energy method that is due to Courant, Friedrichs, and Lewy [8] and depends upon the derivation of an appropriate set of a priori energy estimates. In [6] Nishida conjectured that singularities in the first spatial derivatives of the solutions $(v,w)$ of the system (S) should develop, in general, in finite time, if one relaxes the assumption that the gradients of the initial data $(v,(x,0)\ ,\ w(x,0))$ be small; this conjecture of finite-time breakdown of smooth, i.e., $C^1$ solutions $(v,w)$ of the initial value problem associated with the damped quasilinear system (S) , when the gradients of the initial data are no longer sufficiently small, has been proven valid by Slemrod [9], [10], in connection with his

recent work on the instability of steady shearing flows in nonlinear viscoelastic fluids. Before proceeding, however, with a discussion of the viscoelastic model considered by Slemrod in [9] and [10], and its relation to both the quasilinear system (S) and the quasilinear evolution equation

$$\text{(E)} \qquad w_{tt}(x,t) = \sigma(w(x,t))_{xx} - \gamma\, w_t(x,t) \ , \ 0 \leq x \leq L \ , \ t > 0$$

which is the subject matter of the present paper, we digress briefly to delineate some recent results of MacCamy [11], Dafermos and Nohel [12], and this author [13] on a voscoelastic model which is closely related to the nonlinear model considered in [9], [10]; we also comment below on some related work of Nohel [14] on the damped nonhomogeneous quasilinear wave equation associated with (1.1).

The most widely studied model of one-dimensional nonlinear viscoelastic response seems to be the one which was first studied rigorously by MacCamy in [11]; in this model the displacement field $u(x,t)$ satisfies, on $[0,L)\times[0,\infty]$ a one-dimensional nonlinear integrodifferential equation of the form

$$\text{(I)} \qquad u_{tt} = a(0)\ \sigma(u_x)_x - \int_0^t a_\tau(t-\tau)\sigma(u_x)_x\, d\tau + \mathfrak{F}(x,t)$$

and initial and boundary data of the type

$$\text{(1.7)} \qquad \begin{aligned} &u(0,t) = 0 \ , \ u(L,t) = 0 \ , \ t > 0 \\ &u(x,0) = \tilde{u}_0(x) \ , \ u_t(x,0) = \tilde{u}_1(x) \ , \ 0 \leq x \leq L \end{aligned}$$

By employing Riemann invariants and a set of suitably derived a priori energy estimates, MacCamy showed that the above initial-boundary value problem has a unique classical solution for all $t > 0$ when the data term $\mathfrak{F}$ is suitably restricted and the inital data $\tilde{u}_0$ , $\tilde{u}_1$ are sufficiently

small; it is also proven in [11] that the solution is asymptotically stable, i.e. that it tends to zero as $t \to +\infty$. The essential hypotheses in [11] are that $a(t) = a_\infty + A(t)$, $a_\infty > 0$, $A \in L^1(0,\infty)$, $(-1)^k a^{(k)}(t) \geq 0$, $k = 0$, 1, 2, $\sigma(0) = 0$, $\sigma'(\zeta) \geq \epsilon > 0$, and $|\sigma^{(k)}(\zeta)| \leq \bar{\sigma}$, $k = 0, 1, 2$ for all $\zeta \in R^1$, as well as various smoothness assumptions relative to $\sigma$, $\tilde{u}_o$, $\tilde{u}_1$, and $\mathfrak{F}$; the restrictions on $\mathfrak{F}$ take the form of boundedness and growth conditions. Without loss of generality it may be assumed that $a(0) = 1$ in (I). It can be shown that (I) has the equivalent form (see [10], [11], or [12])

$$(1.8) \qquad u_{tt}(x,t) + \frac{\partial}{\partial t} \int_o^t k(t-\tau)u_t(x,\tau)d\tau = \sigma(u_x(x,t))_x + \Phi(x,t)$$

for $x \in [0,L]$, $0 < t < \infty$, where $k(t)$ is the resolvent kernel associated with $\dot{a}(t)$ and $\Phi(x,t)$ is determined by $k(t)$ and $\mathfrak{F}(x,t)$ Clearly (1.8) is also equivalent to

$$(1.8a) \qquad u_{tt}(x,t) + \gamma\, u_t(x,t) = \sigma(u_x(x,t))_x + \tilde{\Phi}$$

where the functional $\tilde{\Phi}(u)$ is given by

$$(1.8b) \qquad \tilde{\Phi}\,(u(x,t)) = \Phi(x,t) - \int_o^t k_t(t-\tau)u_t(x,\tau)\, d\tau$$

and $\gamma = k(0)$; this damped quasilinear wave equation corresponds to the nonhomogeneous version of (1.1) but has the obvious drawback that the non-homogeneous (or forcing) term $\tilde{\Phi}$ depends on the displacement $u$, a problem which is handled in [11] via the establishment of certain a priori estimates for the solution.

J. A. Nohel [14] recently considered the initial-value problem on $R^1$, for damped non-homogeneous wave equations of the form (1.8a), with

$\Phi = \Phi(x,t)$ independent of $u(x,t)$, and extended Nishida's method for the corresponding homogeneous equation (1.1) so as to obtain global existence and uniqueness of smooth solutions whenever the initial data are sufficiently small (in the sense of Nishida [6], delineated above) and the $L^1(0,\infty)$ and $L^\infty(0,\infty)$ norms of $\tilde{\Phi}(t) = \sup_{x \in R^1} |\tilde{\Phi}(x,t)|$ and the $L^\infty(0,\infty)$ norm of $\tilde{\Phi}_1(t) = \sup_{x \in R^1} |\tilde{\Phi}_x(x,t)|$ are sufficiently small as well. In addition, Nohel [14] is able to prove that the unique global smooth solution of the initial-value problem associated with (1.8a) depends continuously on the data $\tilde{u}_0$, $\tilde{u}_1$, and $\tilde{\Phi}$. In more recent work, Dafermos and Nohel [12] have applied an appropriate modification of Matsumura's energy arguments [7] to the standard initial-boundary value problems associated with the one-dimensional nonlinear integrodifferential equation (I) and deduced the existence of a unique globally defined smooth solution which, under appropriate conditions (again, suitably "small" data $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{\Phi}$) decays to zero as $t \to +\infty$; it is to be noted that unlike the arguments in [6], [9] - [10], [11], which are based on Riemann invariants, and hence are strictly limited to one-dimensional problems, the method of Dafermos and Nohel [12] may be extended to problems in two or three (or even higher) dimensions. It should also be noted that both MacCamy [15], using Riemann invariants in conjunction with energy estimates, and Dafermos and Nohel [12], using energy estimates, have treated the parabolic counterpart of (I) which arises in problems of heat flow in nonlinear one-dimensional heat conductors with memory; Dafermos and Nohel [12] also treat a problem of heat flow in a two-dimensional nonlinear heat conductor with memory thus indicating how Matsumura's arguments extend to problems in

higher dimensions.

In his 1975 paper, MacCamy conjectured that the viscous damping mechanism inherent in (I) was too weak to prevent breakdown of global smooth solutions if the data were sufficiently large in an appropriate sense; to the best of this author's knowledge, that conjecture remains open although Slemrod [9], [10] has proven a finite-time breakdown result for a closely related model of nonlinear one dimensional viscoelastic response, a model which leads to a damped ( $\gamma > 0$) homogeneous system of the form (S) as opposed to a damped nonhomogeneous system of the type (1.8a) , (1.8b) (which is, in turn, equivalent to (I) ) . This author has recently derived [16] growth estimates for solutions of the initial boundary value problem corresponding to (I) without making any assumptions about the size of the data; these results are of the following type: Suppose that $a(0) = 1$ , $\mathcal{F} \equiv 0$ , that $\sigma(\zeta) = \Sigma'(\zeta)$ with $\alpha\Sigma'(\zeta) \geq \zeta\Sigma''(\zeta)$ , $\forall \zeta \in R^1$ , and some $\alpha > 2$ and that $\bar{\sigma} > 0$ such that $|\sigma'(\zeta)| < \bar{\sigma}$ , $\forall \zeta \in R^1$ (no sign definiteness assumptions are imposed on the $a^{(k)}(t)$ , $k = 0, 1, 2$ as in [11] and [12]); let $T > 0$ be fixed. Then any sufficiently smooth solution of the initial boundary value problem corresponding to (I) which lies in the class

$$(1.9) \qquad \mathcal{C} = \{u : [0,T) \to H_0^1[0,L] \mid \sup_{[0,T)} \|u\|_{H_0^1} \leq C \}$$

for some real number $C > 0$ must satisfy the quadratic growth estimate

$$(1.10) \qquad \|u\|^2_{L^2} \geq \|\tilde{u}_0\|^2_{L^2} + 2\nu \|\tilde{u}_0\|_{L^2} t + \nu^2 t^2, \ 0 \leq t < T$$

where $\nu > 0$ is an appropriately chosen constant. To be more precise, the growth estimate (1.10) holds for solutions $u(x,t) \in C^2([0,L] \times [0,T]) \cap \mathcal{C}$

with initial data $(\tilde{u}_o , \tilde{u}_1)$ satisfying

(1.11a) $$\int_o^1 \tilde{u}_o(x)\tilde{u}_1(x)dx > \nu \left(\int_o^1 \tilde{u}_o^2(x)dx\right)^{1/2}$$

where

(1.11b) $$\begin{cases} \nu = 2\alpha\delta/(\alpha - 1) \\ \delta = \max\,(E(0)\ ,\ \bar{\sigma}\,\varkappa_T C^2) \\ E(t) = \frac{1}{2}\int_o^L u_t^2(x,t)dx + \int_o^L \Sigma(u_x(x,t))dx \geq 0 \\ \varkappa_T = |\dot{a}(0)|T + (1 - \frac{1}{\alpha}) \sup\limits_{[0,T)} \int_o^T |\dot{a}(t-\tau)|d\tau \\ \qquad + \sqrt{T} \int_o^T \left(\int_o^t \ddot{a}^2(t-\tau)d\tau\right)^{1/2} dt \end{cases}$$

In (1.11b), $E(t) \geq 0$ follows from the fact that our two hypotheses on $\sigma(\zeta)$ imply that $\Sigma(\zeta) \geq 0$ , $\forall \zeta \in R^1$ ; the growth estimate (1.10) applies, of course, to the unique global smooth solutions of the initial-boundary value problem associated with (I) when the initial data $\tilde{u}_o$ , $\tilde{u}_2$ are sufficiently small in the sense of [11] or [12].

While global nonexistence of smooth solutions has not been proven for the viscoelastic model represented by (I) , when the data $\tilde{u}_o$ , $\tilde{u}_1$ , $\mathfrak{F}$ are appropriately large, it has been proven for a related model of nonlinear viscoelastic response. In [9], [10] Slemrod considers steady shearing flows in a nonlinear voscoelastic fluid in which the stress is given as a real-valued, odd, analytic function $\sigma$ of the linear functional $\int_o^\infty e^{-\gamma s}\mathcal{V}_x(x,t-s)ds$ , where $\mathcal{V}(x,t)$ is the velocity field (actually the y-component of the velocity field in a fixed Cartesian coordinate system (x,y,z)) and $\mathcal{V}_x(x,t)$ is the shear rate. Thus, the shearing stress $T^{xy}(t)$ is given by

(1.12) $$T^{xy}(t) = \sigma\left(\int_o^\infty e^{-\gamma\tau}\mathcal{V}_x(x,t-\tau)d\tau\right)$$

and the equation for conservation of linear momentum then yields the evolution equation

(1.13) $\rho\, v_t(x,t) = \sigma\left(\int_0^\infty e^{-\gamma s} v_x(x,t-s)\,ds\right)_x$

where $\rho > 0$ is an (assumed) constant mass density. Associated with (1.13) in [9], [10] are the no-slip boundary conditions

(1.14) $v(0,t) = 0 \quad , \; v(L,t) = V$

where it is assumed that the fluid is confined between two parallel walls of infinite extent at $x = 0$ and $x = L$ with the top wall at $x = L$ moving with velocity $V$. The system (1.13), (1.14) admits the steady rectilinear flow given by $v(x) = Vx/L$ as a solution and, thus, in order to study the stability of the flow against shearing perturbations Slemrod sets $\hat{v}(x,t) = v(x,t) - Vx/L$ in which case the perturbed flow $v(x,t)$ satisfies

(1.15a) $\rho\, \hat{v}_t(x,t) = \sigma\left(\int_0^\infty e^{-\gamma s}\hat{v}_x(x,t-s)\,ds + \frac{V}{\gamma L}\right)_x$

(1.15b) $\hat{v}(0,t) = 0 \; , \; \hat{v}(L,t) = 0$

to which is coupled the prescription of a smooth velocity history, i.e.,

(1.15c) $\hat{v}(x,\tau) = \hat{v}_0(x,\tau) \; , \; -\infty < \tau \le 0$ .

Clearly, (1.15a) can be rewritten as

(1.15a') $\hat{v}_t(x,t) = \hat{\sigma}\left(\int_0^\infty e^{-\gamma s}\, \hat{v}_x(x,t-s)\,ds\right)_x$

with $\hat{\sigma}(0) = 0$ if we set

(1.16) $\hat{\sigma}(\zeta) = \frac{1}{\rho}\left[\sigma\left(\zeta + \frac{V}{\gamma L}\right) - \sigma\left(\frac{V}{\gamma L}\right)\right]$

Slemrod [9], [10] now is able to transform the initial-history boundary value problem (1.15a'), (1.15b), (1.15c) into an initial-boundary value problem for a damped quasilinear system of the form (S) by introducing the new variables

$$(1.17a)\qquad v(x,t) = \int_0^\infty e^{-\gamma s}\hat{\mathcal{V}}_t(x,t-s)ds$$

$$(1.17b)\qquad w(x,t) = \int_0^\infty e^{-\gamma s}\,\hat{\mathcal{V}}_x(x,t-s)ds$$

Integration by parts in (1.17a) yields

$$v(x,t) = \hat{\mathcal{V}}(x,t) - \gamma\int_0^\infty e^{-\gamma s}\,\hat{\mathcal{V}}(x,t-s)ds$$

It is then immediate that $(v,w)$ satisfy

$$(\bar{S})\qquad \begin{cases} w_t = v_x \\ v_t = \hat{\mathcal{V}}_t - \gamma v \equiv \hat{\sigma}(w)_x - \gamma v \end{cases}$$

in other words $(v,w)$ satisfy (S) with $\sigma$ replaced by $\hat{\sigma}$ . In view of (1.15b), (1.15c) and the definitions of $v(x,t)$, $w(x,t)$ we have associated with $(\bar{S})$ the initial and boundary conditions

$$(1.18a)\qquad v(0,t) = 0\ ,\ v(L,t) = 0$$

$$(1.18b)\qquad v(x,0) = \tilde{v}_0(x)\ ,\ w(x,0) = \tilde{w}_0(x)\ .$$

where $\tilde{v}_0$ , $\tilde{w}_0$ result from the insertion of the velocity history $\mathcal{V}_0(x,\tau)$ in (1.17a), (1.17b). In order that the constitutive relation (1.12) be nonlinear it is necessary that $\sigma''(\zeta^*) \neq 0$ for some $\zeta^* \in R^1$ . By choosing the speed of the top wall $V = \gamma L\zeta^*$ it follows that, in addition to $\hat{\sigma}(0) = 0$ ,

we also have $\hat{\sigma}''(0) \neq 0$. But the crucial requirement imposed in [9], [10] on the system $(\bar{S})$ is that it be strictly hyperbolic at least in a neighborhood of the origin, i.e., that $\hat{\sigma}'(\zeta) > 0, \forall \zeta \in R^1$ such that $|\zeta| < \delta$ for some $\delta > 0$. In this case if one defines the Riemann invariants

$$
(1.19) \qquad \begin{aligned} r(x,t) &= v(x,t) + \int_0^{w(x,t)} \sqrt{\hat{\sigma}'(\zeta)}\, d\zeta \\ s(x,t) &= v(x,t) - \int_0^{w(x,t)} \sqrt{\hat{\sigma}'(\zeta)}\, d\zeta \end{aligned}
$$

and assumes that $|r(x,0)|$, $|s(x,0)|$ are sufficiently small it is possible (Slemrod [9], Nishida [6]) to prove that for as long as smooth solutions of $(\bar{S})$ (1.18a,b) exist, $|r(x,t)|$, $|s(x,t)|$ remain small. Thus, if $\hat{\sigma}'(0) > 0$, and $|r(x,0)|$, $|s(x,0)|$ are chosen sufficiently small, it follows that for as long as smooth solutions exist $w(x,t)$ remains uniformly near zero and hence $\hat{\sigma}'(w) > 0$. With the assumptions that $|r(x,0)|$, $|s(x,0)|$ are *sufficiently small* and either $|r_x(x,0)|$ or $|s_x(x,0)|$ is sufficiently large Slemrod [9], [10] is then able to employ a Riemann invariant argument to prove that $C^1$ (in $(x,t)$ ) solutions $(v,w)$ of $(\bar{S})$, (1.18a,b) exist, for at most, a finite time. As Slemrod [9,§4] notes this finite time breakdown result depends crucially on the local hyperbolicity assumption $\hat{\sigma}'(0) > 0$. For example, in the case of a fluid of integral grade three where

$$\sigma(\zeta) = \sigma_1\zeta + \sigma_3\zeta^3$$

if, $\sigma_1 > 0$, $\sigma_3 < 0$ then, clearly, $\sigma'(\zeta) < 0$ for $|\zeta|$ sufficiently large; however, if $|v(x,0)|$, $|w(x,0)|$ are sufficiently small then $|w(x,t)|$ remains small for as long as smooth solutions of $(\bar{S})$, (1.18a,b) exist and

one never has to worry about the case $|\zeta|$ "sufficiently large" as long as

$$(1.20) \qquad \hat{\sigma}'(0) = \sigma'\left(\frac{V}{\gamma L}\right)^2 = \sigma_1 + 3\sigma_3\left(\frac{V}{\gamma L}\right)^2 > 0$$

i.e., as long as $V/\gamma L$ is small. If, on the other hand, $V/\gamma L \geq \sqrt{\sigma_1/3|\sigma_3|}$ then $\hat{\sigma}'(0) \leq 0$ , loss of hyperbolicity results, and, as the author ([9], [10]) notes no conclusions regarding either global existence or nonexistence of solutions can be obtained from the analysis in [9], [10] (2) .

Our aim in this paper will be to try to address the situation vis a vis initial-boundary value problems associated with the damped $(\gamma > 0)$ quasi-linear system (S) when $\sigma'(0) \leq 0$ ; in general, it will be shown that, for a variety of boundary conditions, one can not expect global smooth solutions of (S) to exist when $\sigma'(0) \leq 0$ even if the initial data functions, $\tilde{v}_o(x), \tilde{w}_o(x)$ , and their gradients, are small in magnitude. In addition, we obtain for various initial-boundary value problems associated with (S) , growth estimates for solutions which are valid on the maximal time interval of existence; many of these growth estimates apply to those well-posed problems associated with (S) which are obtained by restricting, as in [6], the initial data to be sufficiently small. Some of our global nonexistence results may also be applied to the nonlinear viscoelastic fluid model considered in [9], [10] if we replace the no-slip boundary conditions (1.14) by

(2) We note, in passing, that $\hat{\mathcal{V}}_t = \hat{\sigma}(w)_x$ , $\hat{\mathcal{V}}_x = v_x + \gamma w$ so that if $(v,w)$ is not of class $C^1$ then the velocity field $\mathcal{V}(x,t)$ is not of class $C^1$ (in $(x,t)$) .

boundary conditions involving both the shear rates at $x = 0$ and $x = L$, or the shear rate at $x = L$ and its gradient at $x = 0$, and work with the flow $\mathcal{V}(x,t)$ directly instead of with shearing perturbations of a steady flow.

Our results also cover certain situations where $\hat{\sigma}'(0) > 0$ but $\sigma'(\zeta) \leq 0$ for $|\zeta|$ sufficiently large. In the case of the fluid of grade three, for example, i.e., $\sigma(\zeta) = \sigma_1\zeta + \sigma_3\zeta^3$, $\sigma_1 > 0$, $\sigma_3 < 0$, $\sigma'(\zeta) < 0$ if $|\zeta| > \sigma_1/3|\sigma_3|$. If the initial data $\tilde{v}_0(x)$, $\tilde{w}_0(x)$ are sufficiently small then, with $\sigma'(0) = \sigma_1 > 0$, it is guaranteed, by the results of Nishida [6] and Slemrod [9], that $|w(x,t)|$ remains small and, in fact, smaller than $\sigma_1/3|\sigma_3|$, for as long as smooth solutions of (S) exist; in this case $\sigma'(w(x,t)) > 0$ on the maximal time interval of existence. On the other hand if $|v(x,0)|$, $|w(x,0)|$ are not sufficiently small then there is no guarantee that $|w(x,t)|$ remains smaller than the critical value $\sigma_1/3|\sigma_3|$ in which case there may be values of $(x,t)$ such that $\sigma'(w(x,t)) < 0$; the global nonexistence result of [9], [10] do not seem to cover this possibility either.

Our approach to the quasilinear system (S) shall be through the equivalent damped quasilinear wave equation (E). That (S) and (E) are equivalent is easily established, i.e., if $(v,w)$ is a solution of (S) then by elimination between the first and second equations which comprise (S) it follows that $w(x,t)$ satisfies (E). On the other hand, if $w(x,t)$ satisfies (E) we may multiply (E) through by $e^{\gamma t}$ and obtain the fact that $w(x,t)$ satisfies

$$(e^{\gamma t}w_t(x,t))_t = (e^{\gamma t}\sigma(w(x,t)_x)_x$$

which (at least in a simply connected domain of $(x,t)$ space) implies that $\exists q(x,t)$ such that

$$q_x(x,t) = e^{\gamma t}w_t(x,t) \ , \ q_t(x,t) = e^{\gamma t}\sigma(w(x,t))_x$$

If we set $v(x,t) = e^{-\gamma t}q(x,t)$ it then follows directly that $v_x = w_t$ and $v_t = \sigma(w)_x - \gamma v$ , i.e., that $(v,w)$ satisfies (S) . Actually, given that $w(x,t)$ is a solution of (E) one may construct a function $v(x,t)$ , such that $(v,w)$ is a solution of (S) by simply integrating the first equation in (S) w.r.t. $x$ to obtain

$$v(x,t) = \int_0^x w_t(y,t)dy + f(t)$$

and then substituting into the second equation in (S) and replacing the resulting term $w_{tt}(y,t)$ by $\sigma(w(y,t))_{yy} - \gamma w_t(y,t)$ ; in this manner, one easily obtains $v(x,t)$ as

$$v(x,t) = \int_0^x w_t(y,t)dy + e^{-\gamma t}\int_0^t e^{\gamma\tau}\sigma'(w(0,\tau))w_x(0,\tau)d\tau \tag{1.2}$$

to within an arbitrary constant of integration. The pair $(v,w)$ is then a solution of (S) . If growth estimates for solutions of initial-boundary value problems associated with (E) can be obtained then (1.21) can, in principle, be used to derive growth estimates for the corresponding $v(x,t)$ which is such that the pair $(v,w)$ is a solution of an equivalent initial-boundary value problem associated with (S) .

Through the remainder of the paper we shall assume that $\sigma(\zeta)$ is of class $C^2(R^1)$, and genuinely nonlinear, so that $\exists\zeta^* \in R^1$ for which $\sigma''(\zeta^*) \neq 0$ .

In addition, we shall confine our attention to nonlinearities $\sigma(\zeta)$ that satisfy a specific growth condition which is delineated in §2 ; this restriction essentially limits the class of nonlinearities $\sigma(\zeta)$ to those which are such that the absolute value of the indefinite integral of $\sigma(\zeta)$ grows slower than a polynomial in $|\zeta|$ of degree $\alpha$ , for an appropriate $\alpha > 0$ .

Finally, by a <u>regular</u> solution $w(x,t)$ of an initial-boundary value problem associated with (E), with homogeneous boundary data $w(0,t) = w(L,t) = 0$ , we shall understand, in the sequel, a solution $w \in C^2((0,L)\times[0,\infty))$ such that $w_x(0,t) = \lim_{y\to 0^+} \left( \frac{\partial w(y,t)}{\partial y} \right) < +\infty$ with

(1.22) $\quad w_x^2(0,\,\cdot\,) \in L^\infty[0,\infty) \cap L^1[0,\infty)$

The motivation for this definition of regular solution will be clear from the analysis in §2 ; for initial-boundary value problems for (E) with homogeneous boundary data $w_x(0,t) = 0$ , $w(L,t) = 0$ , the definition of <u>regular</u> solution will be modified to mean a solution $w \in C^2((0,L) \times [0,\infty))$ such that for $0 \le t < \infty$ ,

(1.23) $\quad \sigma'(w(0,t)) = \lim_{y\to 0^+} \sigma'(w(y,t)) < +\infty$

2. Some Basic Estimates

In this section we will derive an energy estimate as well as several other estimates which are satisfied by a particular real-valued functional $\mathfrak{F}(w(x,t))$ which is defined on regular soultions of initial-boundary value problems associated with the quasilinear evolution equation (E) ; these estimates will be used in §3 to prove nonexistence of regular solutions, i.e., global nonexistence of sufficiently smooth solutions on $[0,\infty)$ , as well as to derive various growth estimates (lower bounds) which are valid on the maximal interval of existence $[0,t_{max})$ , $t_{max} \leq +\infty$ , of a sufficiently smooth solution $w(x,t)$; without loss of generality we may take $L = 1$ in all that follows. Thus, let $w(x,t)$ be a regular solution of (E) with associated inital and boundary data of the form

$$(2.1) \qquad w(x,0) = \tilde{w}_0(x) \ , \ w_t(x,0) = \tilde{w}_1(x) \ , \ 0 \leq x \leq 1$$

$$(2.2) \qquad w(0,t) = 0 \ , \ w(1,t) = 0 \ , \ t > 0$$

where $\tilde{w}_0(\cdot)$ , $\tilde{w}_1(\cdot)$ are assumed to be of class $C^2$ on $[0,1]$ .

Concerning the nonlinearity $\sigma(\zeta)$ in (E) , we will assume, in addition to the hypotheses delineated in §1 , that $\sigma(0) = 0$ and that $\Sigma(\zeta) = \int_0^\zeta \sigma(\rho)d\rho$, $\zeta \in R^1$ , satisfies

$$(2.3) \qquad \alpha\,\Sigma(\zeta) \geq \zeta\,\Sigma'(\zeta) \ , \ \forall \zeta \in R^1 \text{ and some } \alpha > 0 \ .$$

In the sequel our global nonexistence theorems and growth estimates will hold for $\sigma(\zeta)$ such that $\Sigma(\zeta)$ satisfies (2.3) for $\alpha$ in an appropriately chosen interval of $(0,\infty)$ . For $\sigma(\zeta) = \sigma_1\zeta + \sigma_3\zeta^3$ it is easily seen that (2.3) is equivalent to the statement that

$$(2.4) \qquad \sigma_1(\tfrac{\alpha}{2} - 1)\cdot \zeta^2 + \sigma_3\ (\tfrac{\alpha}{4} - 1)\ \zeta^4 \geq 0\ ,\ \forall \zeta \in R^1$$

If $\sigma_1 > 0$ , $\sigma_3 > 0$ , (2.4) is satisfied for any $\alpha \geq 4$ . If $\sigma_1 < 0$ , $\sigma_3 < 0$ then (2.4) is satisfied for any $\alpha$ , $0 < \alpha < 2$ . With $\sigma_1 > 0$ , $\sigma_3 < 0$ , (2.4) is satisfied with $2 \leq \alpha \leq 4$ while if $\sigma_1 < 0$ , $\sigma_3 > 0$ (2.4) is not satisfied by any $\alpha$ for all $\zeta \in R^1$ . The example of $\sigma(\zeta) = \sigma_1\zeta + \sigma_3\zeta^3$ will be used in several places later on in the paper. The inequality is a restriction on the growth of $\Sigma(\zeta)$ in the sense that it implies that $\forall \zeta \in R^1$ , $|\Sigma(\zeta)| \leq C\ |\zeta|^\alpha$ for some $C > 0$ , $\alpha > 0$ . The additional restriction that $|\sigma'(\zeta)| < \bar{\sigma}$ , for some $\bar{\sigma} > 0$ , would imply that $\Sigma(\zeta) \geq 0$ , $\forall \zeta \in R^1$ ; this hypothesis was employed in [13] but will not be used in this paper and, in fact, we want to allow for $\sigma(\zeta)$ which are such that $\Sigma(\zeta) < 0$ for $|\zeta|$ sufficiently large.

For a regular solution $w(x,t)$ of (E) , (2.1) , (2.2) we define an energy functional

$$(2.5a) \qquad E(t) = \frac{1}{2} \int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx + \int_0^1 \Sigma(w(x,t))dx$$

and set

$$(2.5b) \qquad \tilde{E}(t) = E(t) - \left( \frac{\sigma'^2(0)}{4\gamma} \right) \int_0^t w_x^2(0,\tau)d\tau$$

We claim that $\tilde{E}(t) \leq \tilde{E}(0) = E(0)$ , for all $t$ , $0 \leq t < \infty$ .
In order to demonstrate this we compute

$$(2.6) \qquad \dot{E}(t) = \int_0^1 \left(\int_0^x w_t(y,t)dy\right)\left(\int_0^x w_{tt}(y,t)dy\right)dx$$
$$+ \int_0^1 \Sigma'(w(x,t))w_t(x,t)dx$$

$$= \int_0^1 \left(\int_0^x w_t(y,t)dy\right)\left(\int_0^x (\sigma(w(y,t)))_{yy} - \gamma w_t(y,t))dy\right)dx + \int_0^1 \Sigma'(w(x,t))w_t(x,t)dx$$

$$= \int_0^1 \left(\int_0^x w_t(y,t)dy\right) \sigma(w(y,t)),_y \int_0^x dx - \gamma \int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx + \int_0^1 \Sigma'(w(x,t))w_t(x,t)dx$$

The result is also true on $[0,t_{max})$ for solutions which are of class $C^2$ on $([0,L] \times [0,t_{max}))$, satisfying (1.22a-c), where $[0,t_{max})$ , $t_{max} \leq \infty$ , is the maximal interval of existence. Thus

$$\dot{E}(t) = \int_0^1 \frac{\partial}{\partial x}\left[\left(\int_0^x w_t(y,t)dy\right)\sigma(w(x,t))\right]dx - \int_0^1 w_t(x,t)\sigma(w(x,t))dx + \int_0^1 \Sigma'(w(x,t))w_t(x,t)dx - \sigma'(w(0,t))w_x(0,t) \int_0^1 \left(\int_0^x w_t(y,t)dy\right)dx - \gamma \int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx$$

$$= -\sigma'(0)w_x(0,t) \int_0^1 \left(\int_0^x w_t(y,t)dy\right)dx - \gamma \int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx$$

as $\sigma(w(1,t)) = \sigma(0) = 0$ and $\Sigma'(\zeta) = \sigma(\zeta)$ , $\forall \zeta \in R^1$ . If we set $\lambda(t) = \int_0^1 \left(\int_0^x w_t(y,t)dy\right)dx$ , then we have (Cauchy-Schwarz Inequality)

$$\lambda^2(t) \leq \int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx$$

and

$$-\dot{E}(t) \geq [\sigma'(0)w_x(0,t)\lambda(t) + \gamma\lambda^2(t)] = \gamma\left[\lambda^2(t) + \left(\frac{\sigma'(0)w_x(0,t)}{\gamma}\right)\lambda(t)\right] \tag{2.7}$$

$$= \gamma[\lambda(t) + (\frac{\sigma'(0)w_x(0,t)}{2\gamma})]^2$$

$$- \gamma(\frac{\sigma'(0)w_x(0,t)}{\gamma})^2$$

Therefore, dropping the nonnegative term on the right-hand side of (2.7) we have

$$\dot{E}(t) \le \frac{\sigma'^2(0)}{4\gamma} w_x^2(0,t) \tag{2.8}$$

or, for $0 \le t < \infty$ ,

$$E(t) \le E(0) + \frac{\sigma'^2(0)}{4\gamma} \int_0^t w_x^2(0,\tau)d\tau \tag{2.9}$$

Defining $\tilde{E}(t) = E(t) - \frac{\sigma'^2(0)}{4\gamma}\int_0^t w_x^2(0,\tau)d\tau$ we see that $\tilde{E}(t) \le E(0) = \tilde{E}(0)$ . We state our result as

Lemma 1: Let $w(x,t)$ be a regular solution of (E) , (2.1), (2.2), and define

$$\tilde{E}(t) = \frac{1}{2}\int_0^1 (\int_0^x w_t(y,t)dy)^2 dx$$
$$+ \int_0^1\int_0^{w(x,t)} \sigma(\rho)d\rho dx - \frac{\sigma'^2(0)}{4\gamma}\int_0^t w_x^2(0,\tau)d\tau$$

Then, $\tilde{E}(t) \le \tilde{E}(0)$ for all $t$ , $0 \le t < \infty$ .

We now introduce the real-valued function $\mathfrak{F}(t) = \mathfrak{F}(w(\cdot,t))$ which is defined, for $0 \le t < \infty$ , on regular solutions $w(x,t)$ of (E) , (2.1), (2.2) by

$$\mathfrak{F}(t) = \int_0^1(\int_0^x w(y,t)dy)^2 dx + \beta(t + t_o)^2 \tag{2.10}$$

where $\beta_o \ge 0$ , $t_o \ge 0$ are arbitrary; we will obtain several lower bounds on the derivative $\mathfrak{F}''(t)$ that will be used in the sequel to derive various second order differential inequalities which are satisfied by $\mathfrak{F}(t)$ when

$w(x,t)$ is a regular solution of (E), (2.1), (2.2). We begin by computing directly

(2.11) $$\mathfrak{F}'(t) = 2\int_0^1 \left(\int_0^x w(y,t)dy\right)\left(\int_0^x w_t(y,t)dy\right)dx + 2\beta(t+t_0)$$

and

(2.12) $$\begin{aligned}\mathfrak{F}''(t) &= 2\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 \\ &\quad + 2\int_0^1\left(\int_0^x w(y,t)dy\right)\left(\int_0^x w_{tt}(y,t)dy\right)dx + 2\beta \\ &= 2\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx - 2\gamma\int_0^1 \left(\int_0^x w(y,t)dy\right)\left(\int_0^x w_t(y,t)dy\right)dx \\ &\quad + 2\int_0^1 \left(\int_0^x w(y,t)dy\right)\left[\left(\sigma(w(y,t))_y \Big|_0^x\right)\right]dx + 2\beta \\ &= 2\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx \\ &\quad + 2\int_0^1 \left(\int_0^x w(y,t)dy\right)\sigma(w(x,t))_x dx \\ &\quad - 2\sigma'(0)w_x(0,t)\int_0^1 \left(\int_0^x w(y,t)dy\right)dx \\ &\quad - \gamma\left(\mathfrak{F}'(t) - 2\beta(t+t_0)\right)) + 2\beta\end{aligned}$$

where we have used (2.11). Thus,

(2.13) $$\begin{aligned}\mathfrak{F}''(t) &\geq -\gamma\mathfrak{F}'(t) + 2\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx \\ &\quad + 2\int_0^1 \left(\int_0^x w(y,t)dy\right)\sigma(w(x,t))_x dx \\ &\quad - 2\sigma'(0)w_x(0,t)\int_0^1 \left(\int_0^x w(y,t)dy\right)dx + 2\beta\end{aligned}$$

On the other hand,

(2.14) $$\begin{aligned}H(t) &\equiv -2\gamma\int_0^1 \left(\int_0^x w(y,t)dy\right)\left(\int_0^x w_t(y,t)dy\right)dx \\ &\geq -\gamma\int_0^1 \left[\left(\int_0^x w(y,t)dy\right)^2 + \left(\int_0^x w_t(y,t)dy\right)^2\right] dx \\ &\geq -\nu\left[\mathfrak{F}(t) - \beta(t+t_0)^2\right] - \gamma\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx \\ &\geq -\gamma\,\mathfrak{F}(t) - \gamma\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx\end{aligned}$$

Using this estimate in ($2.12_2$) we have

$$(2.15) \qquad \mathcal{F}''(t) \geq -\gamma\mathcal{F}(t) + (2-\gamma)\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx$$
$$+ 2\int_0^1 \left(\int_0^x w(y,t)dy\right)\sigma(w(x,t))_x dx$$
$$- 2\sigma'(0)w_x(0,t)\int_0^1 \left(\int_0^x w(y,t)dy\right)dx + 2\beta \quad .$$

We will now simplify the lower bounds (2.13), (2.15) for $\mathcal{F}''(t)$ by making use of our hypotheses regarding $\sigma(\zeta), \zeta \in R^1$ , our definition of a regular solution, and the estimate given by Lemma 1; we begin with (2.15), integrating the third term on the right-hand side of the estimate by parts so as to obtain

$$(2.16) \qquad \mathcal{F}''(t) \geq - \gamma\, \mathcal{F}(t) + (2 - \gamma) \int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx$$
$$- 2\, \sigma'(0)\, w_x(0,t) \int_0^1 \left(\int_0^x w(y,t)dy\right)dx$$
$$- 2 \int_0^1 w(x,t)\sigma(w(x,t))dx + 2\beta$$

where we have again used the fact that $\sigma(w(1,t)) = 0$ , $0 \leq t < \infty$ . By adding and subtracting the term $2\alpha \int_0^1 \Sigma(w(x,t))dx$ in (2.16) we then obtain

$$(2.17) \qquad \mathcal{F}''(t) \geq - \gamma\, \mathcal{F}(t) + (2 - \gamma) \int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx$$
$$- 2\alpha \int_0^1 \Sigma(w(x,t))dx$$
$$- 2\, \sigma'(0)w_x(0,t) \int_0^1 \left(\int_0^x w(y,t)dy\right)dx + 2\beta$$
$$+ 2 \int_0^1 (\alpha\, \Sigma\, (w(x,t)) - w(x,t)\Sigma'(w(x,t)))dx$$

Now, as per our hypothesis relative to $\sigma(\zeta)$ , $\alpha\Sigma(\zeta) \geq \zeta\Sigma'(\zeta)$ , $\forall \zeta \in R^1$ , for some $\alpha > 0$ ; choosing $\alpha$ in (2.17) sufficiently large[3] (i.e., restricting the growth of $\Sigma(\zeta)$ so that $|\Sigma(\zeta)| \leq C|\zeta|^\alpha$ , $\forall \zeta \in R^1$) it follows that we may drop the last integral on the right-hand side of (2.17). Also, by the first lemma,

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(3) Sufficiently large means we will later have to restrict $\alpha$ so that $\alpha_1 < \alpha < \alpha_2$ where $\alpha_1 \geq 0$ , $\alpha_2 \leq \infty$. This further narrows the class of admissible nonlinearities $\sigma(\zeta)$ .

$$(2.18)\qquad 2\alpha\int_0^1 \Sigma(w(x,t))dx = 2\alpha\tilde{E}(t) - \alpha\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx + \frac{\alpha\sigma'^2(0)}{2\gamma}\int_0^t w_x^2(0,\tau)d\tau$$

or

$$(2.19)\qquad -2\alpha\int_0^1 \Sigma(w(x,t))dx \geq -2\alpha E(0) + \alpha\int_0^t \left(\int_0^x w_t(y,t)dy\right)^2 dx - \frac{\alpha\sigma'^2(0)}{2\gamma}\int_0^t w_x^2(0,\tau)d\tau$$

and, therefore, (2.17) yields the lower bound

$$\begin{aligned}
(2.20)\qquad \mathcal{F}''(t) \geq & -\gamma\,\mathcal{F}(t) + (2+\alpha-\gamma)\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx \\
& - \frac{\alpha\sigma'^2(0)}{2\gamma}\int_0^t w_x^2(0,\tau)d\tau \\
& - 2\sigma'(0)w_x(0,t)\int_0^1 \left(\int_0^x w(y,t)dy\right)dx \\
& - 2\alpha E(0) + 2\beta \\
\geq & -\gamma\,\mathcal{F}(t) + (2+\alpha-\gamma)\left[\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx + \beta\right] \\
& - \frac{\alpha\sigma'^2(0)}{2\gamma}\int_0^t w_x^2(0,\tau)d\tau \\
& - 2\sigma'(0)w_x(0,t)\int_0^1 \left(\int_0^x w(y,t)dy\right)dx \\
& - \alpha\,[\beta + 2E(0)]
\end{aligned}$$

where we have dropped a term $\gamma\beta \geq 0$ on the right-hand side of the last estimate.

Now,

$$\begin{aligned}
& 2\sigma'(0)w_x(0,t)\int_0^1 \left(\int_0^x w(y,t)dy\right)dx \\
& \leq 2\sigma'^2(0)\,w_x^2(0,t) + \frac{1}{2}\int_0^1 \left(\int_0^x w(y,t)dy\right)^2 dx \\
& \leq 2\sigma'^2(0)\,w_x^2(0,t) + \frac{1}{2}\left(\mathcal{F}(t) - \beta(t+t_0)^2\right)
\end{aligned}$$

so that

$$- 2\sigma'(0)\, w_x(0,t) \int_0^t \left(\int_0^x w(y,t)dy\right)dx \geq$$

$$- 2\sigma'^2(0) w_x^2(0,t) - \frac{1}{2}\, \mathcal{F}(t)$$

Using this result in $(2.20_2)$ we obtain

$$(2.21) \qquad \mathcal{F}''(t) \geq -\left(\nu + \frac{1}{2}\right) \mathcal{F}(t) + (2 + \alpha - \gamma)\left[\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx + \beta\right]$$

$$- \sigma'^2(0)\left[2w_x^2(0,t) + \frac{\alpha}{2\gamma} \int_0^t w_x^2(0,\tau)d\tau\right]$$

$$- \alpha[\beta + 2\, E(0)]$$

*However, as* $w(x,t)$ *is assumed to be a regular solution of* (E), (2.1), (2.2) we have

$$(2.22) \qquad \mathcal{F}''(t) \geq -\left(\gamma + \frac{1}{2}\right) \mathcal{F}(t) + (2 + \alpha - \gamma)\left[\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx + \beta\right]$$

$$- \sigma'^2(0)\left[2\|w_x^2(0,\cdot)\|_{L^\infty} + \frac{\alpha}{2\gamma}\|w_x^2(0,\cdot)\|_{L^1}\right]$$

$$- \alpha\,[\beta + 2E(0)] \quad .$$

Setting,

$$\kappa_\infty = \frac{2}{\alpha}\, \|w_x^2(0,\cdot)\|_{L^\infty} + \frac{1}{2\gamma}\, \|w_x^2(0,\cdot)\|_{L^1}$$

we have, as our final estimate here

$$(2.23) \qquad \mathcal{F}''(t) \geq -(\gamma + 1)\mathcal{F}(t) + (2 + \alpha - \gamma)\left[\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx + \beta\right]$$

$$- \alpha\,[\beta + \kappa_\infty \sigma'^2(0) + 2E(0)]$$

and this completes the reduction of (2.15).

We now turn to (2.13) and work on this lower bound in an analogous fashion, i.e., integrating the third term on the right-hand side of (2.13) by parts we obtain

$$(2.24) \qquad \mathcal{F}''(t) \geq -\gamma\, \mathcal{F}'(t) + 2\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx$$

$$- 2\int_0^1 w(x,t)\sigma(w(x,t))dx$$

$$- 2\sigma'(0)w_x(0,t) \int_0^1 \left(\int_0^x w(y,t)dy\right)dx + 2\beta$$

We again add and substract $2\alpha \int_0^1 \Sigma(w(x,t))dx$ (on the right-hand side of (2.24), employ our growth hypothesis relative to $\Sigma(\zeta)$, $\zeta \epsilon R^1$, and use the first (energy) lemma so as to obtain the lower bound

$$(2.25) \qquad \mathcal{F}''(t) \geq -\gamma \mathcal{F}'(t) + (2+\alpha)\left[\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx + \beta\right]$$
$$- \alpha\,[\beta + 2E(0)]$$
$$- \frac{\alpha\sigma'^2(0)}{2\gamma} \int_0^t w_x^2(o,\tau)d\tau$$
$$- 2\sigma'(0)w_x(0,t) \int_0^1 \left(\int_0^x w(y,t)dy\right)dx$$

where we have added and substracted the term $\alpha\beta \geq 0$ . Estimating the last expression on the right-hand side of (2.25), as per the discussion preceding (2.2), we obtain, *in place of* (2.23),

$$(2.26) \qquad \mathcal{F}''(t) \geq -\gamma \mathcal{F}'(t) - \frac{1}{2}\mathcal{F}(t)$$
$$+ (2+\alpha)\left[\int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx + \beta\right]$$
$$- \alpha\,[\beta + \varkappa_\infty \sigma'^2(0) + 2E(0)]$$

We may, therefore, state

<u>Lemma 2</u>. If $w(x,t)$ is a regular solution of (E), (2.1), (2.2) and $\sigma(\zeta)$, $\zeta \epsilon R^1$, satisfies (2.3) then $\mathcal{F}(t)$, as given by (2.10), with $\beta \geq 0$, $t_o \geq 0$ arbitrary, has a second derivative $\mathcal{F}''(t)$ which possesses the lower bounds given by (2.23) and (2.26).

We also have, directly form (2.25)

<u>Lemma 3</u>: If $w(x,t)$ is a regular solution of (E), (2.1), (2.2) and $\sigma(\zeta)$, $\zeta \epsilon R^1$, satisfies (2.3) then $\mathcal{F}(t)$, as given by (2.10), with $\beta \geq 0$, $t_o \geq 0$ satisfies

(2.27) $\mathfrak{J}''(t) \geq \gamma\, \mathfrak{J}'(t) + (2+\alpha)\, [\int_0^1 (\int_0^x w_t(y,t)dy)^2 dx + \beta]$

$- \alpha\, [\beta + 2E(0)]$

whenever $\sigma'(0) = 0$ .

Now, suppose that we replace the boundary conditions (2.2) by

(2.2') $w_x(0.t) = 0\ ,\ w(1,t) = 0\ ,\ t > 0$ .

In the proof of the energy lemma (lemma 1) we would then obtain, in place of (2.65),

(2.6') $\dot{E}(t) = - \sigma'(w(0,t))w_x(0,t) \int_0^1 (\int_0^x w_t(y,t)dy)dx$

$- \gamma \int_0^1 (\int_0^x w_t(y,t)dy)^2 dx$

$= - \gamma \int_0^1 (\int_0^x w_t(y,t)dy)^2 dx < 0$

provided we modify the definition of a regular solution to be such that $\sigma'(w(0,t)) < +\infty$ , $t \geq 0$ . By (2.6'), $E(t) \leq E(0)$ , $0 \leq t < \infty$ . Also, whenever the expression $\sigma'(w(0,t))$ appears, it always appears in conjunction with the boundary term $w_x(o,t)$ as $\sigma'(w(0,t))w_x(0,t) \equiv 0$ by (2.2') and the assumption that $\sigma'(w(0,t))$ is finite for $0 \leq t < \infty$ . Repeating the analysis that led to the estimate (2.25), and defining a regular solution of (E), (2.1), (2.2') to be a solution $w \in C^2((0,1) \times [0,\infty])$ such that for $0 \leq t < \infty$

(2.28) $\sigma'(w(0,t)) = \lim_{y \to 0^+} \sigma'(w(y,t)) < +\infty$

we may state

<u>Lemma 4</u>. If $w(x,t)$ is a regular solution of (E), (2.1), (2.2') and $\sigma(\zeta)$, $\forall \zeta \in R^1$ , satisfies (2.3), then $\mathfrak{J}(t)$ , as given by (2.10), with $\beta \geq 0$ , $t_o \geq 0$ satisfies (2.27).

<u>Remarks</u>. The lower bounds represented by (2.23), (2.26) will lead to growth estimates for smooth solutions $w(x,t)$ of (E), (2.1), (2.2) which are valid on the maximal interval of existence $[0, t_{max})$, $t_{max} \leq +\infty$; the lower bound represented by (2.27) will, on the other hand, yield the assertion that, with appropriate assumptions concerning the initial data, regular solutions of (E), (2.1), (2.2) can not exist whenever $\sigma'(0) = 0$ and that globally defined regular solutions of (E), (2.1), (2.2') can not exist (with "regular" interpreted in the appropriate sense for each of the respective initial-boundary value problems). We have been unable to prove that a lower bound like (2.27) is valid for regular solutions of (E), (2.1), (2.2) when $\sigma'(0) \neq 0$ although we conjecture that such a lower bound applies in this situation also.

## 3. Growth Estimates and Global Nonexistence Theorems

In this section we will indicate how the lower bounds on $\mathcal{F}''(t)$ derived in the last section lead to global nonexistence theorems and growth estimates for the appropriately defined regular solutions of (E), (2.1), (2.2) and (E), (2.1), (2.2'). Our results then carry over to solutions of the corresponding initial-boundary value problems associated with the system (S): we will also comment on the implications of our global nonexistence theorems for initial-history boundary value problems that can be associated with the nonlinear viscoelastic model defined by (1.13) when either $\sigma'(0) = 0$ or $\sigma'(\zeta) < 0$ for $|\zeta|$ sufficiently large.

We begin with the case in which $w(x,t)$ is assumed to be a regular solution of (E), (2.1), (2.2) with $\sigma(\zeta)$ satisfying $\sigma'(0) = 0$ and (2.3), $\zeta \in R^1$ ; in this situation (2.27) has been shown to be applicable where $\mathcal{F}(t)$ is defined by (2.10), $\beta \geq 0$ , $t_o \geq 0$ are arbitrary and

$$E(0) = \frac{1}{2}\int_o^1 \left(\int_o^x \tilde{w}_1(y)dy\right)^2 dx + \int_o^1 \Sigma(\tilde{w}_o(x))dx$$

By (2.11)

$$\begin{aligned}\mathcal{F}'(t)^2 &\leq 4\left[\int_o^1 \left(\int_o^x w(y,t)dy\right)\left(\int_o^x w_t(y,t)dy\right)dx + \beta(t+t_o)\right]^2 \\ &\leq 8\int_o^1 \left(\int_o^x w(y,t)dy\right)^2 dx \int_o^1 \left(\int_o^x w_t(y,t)dy\right)^2 dx \\ &\quad + 8\beta^2(t+t_o)^2\end{aligned}$$

Therefore, by the Schwartz inequality

$$\begin{aligned}\mathcal{F}\mathcal{F}'' - \frac{(\alpha+2)}{8}\mathcal{F}'^2 \geq &- \gamma\,\mathcal{F}\mathcal{F}' - \alpha\,\mathcal{F}(\beta + 2E(0)) \\ &+ (\alpha+2)\Big\{\left[\int_o^1 \left(\int_o^x w(y,t)dy\right)^2 dx + \beta(t+t_o)^2\right] \\ &\times\left[\int_o^1 \left(\int_o^x w_t(y,t)dy\right)^2 dx + \beta\right] \\ &- \Big(\int_o^1 \left(\int_o^x w(y,t)dy\right)^2 dx \int_o^1 \left(\int_o^x w_t(y,t)dy\right)^2 dx \\ &\qquad + \beta^2(t+t_o)^2\Big)\Big\}\end{aligned}$$

$$\geq -\gamma \mathfrak{F}\mathfrak{F}' - \alpha\mathfrak{F}(\beta + 2E(0))$$

We now set $\mu = \frac{\alpha-6}{8}$ and restrict our attention to those nonlinearities $\sigma(\zeta)$, $\zeta \in R^1$, for which (2.3) holds with $\alpha > 6$. (In those cases where we may take $\beta = 0$ in the analysis, repitiion of the above argument shows that

$$\mathfrak{F}_o\mathfrak{F}_o'' - \frac{(\alpha+2)}{4}\mathfrak{F}_o'^2 \geq -\gamma \mathfrak{F}_o\mathfrak{F}_o' - 2\alpha E(0)\mathfrak{F}_o$$

applies with $\mathfrak{F}_o(t) = \int_0^1 (\int_0^x w(y,t)dy)^2 dx$ ; in this case we set $\mu = \frac{\alpha-2}{4}$ and require that $\sigma(\zeta)$, $\zeta \in R^1$, satisfy (2.3) with $\alpha > 2$). In either case we have, therefore

(3.1) $\quad \mathfrak{F}\mathfrak{F}'' - (\mu+1)\mathfrak{F}'^2 \geq \gamma \mathfrak{F}\mathfrak{F}' - \alpha\mathfrak{F}(\beta + 2E(0))$

with $\mu > 0$. We now define the quantities

$$\mathcal{J}(\xi) = \int_0^1 (\int_0^x \xi(y)dy)^2 dx$$
$$\mathcal{J}(\xi,\lambda) = \int_0^1 (\int_0^x \xi(y)dy)(\int_0^x \lambda(y)dy)dx$$

and consider, first, the case in which the initial data satisfy

(3.2) $\quad \mathcal{J}(\tilde{w}_o, \tilde{w}_1) > 0$

$\qquad E(0) \leq 0$

In this case we may take $\beta = 0$ in (3.1), $\mu = \frac{\alpha-2}{4}$, and assume that $\sigma(\zeta)$ satisfies (2.3) for some $\alpha > 2$. Then (3.1) reduces to

(3.3) $\quad \mathfrak{F}_o\mathfrak{F}_o'' - (\mu+1)\mathfrak{F}_o'^2 \geq -\gamma\mathfrak{F}_o\mathfrak{F}_o'$, $0 \leq t < \infty$

$\mathfrak{F}_o(t) = \int_0^1 (\int_0^x w(y,t)dy)^2 dx$, with $w(x,t)$ a regular solution of (E), (2.1), (2.2). However, (3.3) is equivalent to the differential inequality

(3.4) $\quad [e^{\gamma t}(\mathfrak{F}_o^{-\mu}(t))']' \leq 0$, $0 \leq t < \infty$

Direct integration of (3.4) then yields the estimate

$$(3.5)\qquad \mathcal{F}_o^{\mu}(t) \geq \mathcal{F}_o^{\mu}(0)\,[1 - (1-e^{-\gamma t})\,\frac{\mu\mathcal{F}'(0)}{\gamma\mathcal{F}(0)}\,]^{-1}$$

However, the expression in the brackets on the right-hand side of this estimate will vanish at

$$t = t_\infty \equiv \ln\,[1 - (\frac{\gamma}{2\mu})\,\frac{\mathcal{J}(\tilde{w}_o)}{\mathcal{K}(\tilde{w}_o,\,\tilde{w}_1)}\,]^{-1/\gamma} > 0$$

provided that $\mathcal{K}(\tilde{w}_o, \tilde{w}_1) > (\gamma/2\mu)\,\mathcal{J}(\tilde{w}_o)$. It thus follows that regular solutions of (E), (2.1), (2.2) can not exist, i.e. there can not exist a solution $w(x,t)$ of (E), (2.1), (2.2) which is such that

$$w \in C^2([0,1]\times[0,\infty])\ ,\ \lim_{y\to 0}\ (\frac{\partial w(y,t)}{\partial y}) < +\infty$$

with $w_x^2(0;\cdot) \in L^\infty[0,\infty) \cap L^1[0,\infty)$ .

<u>Example</u> Take $\sigma(\zeta) = \sigma_3\zeta^3$ ; then $\sigma'(0) = 0$ and $\sigma(\zeta)$ , $\zeta \in R^1$ , satisfies (2.3) with $\alpha \geq 4$ if $\sigma_3 > 0$ . If $\sigma_3 < 0$ then (2.3) is satisfied only for $0 \leq \alpha \leq 4$ : the global nonexistence result above, however, only applies to those nonlinearities $\sigma(\zeta)$ for which (2.3) is satisfied, $\forall \zeta \in R^1$ , for some $\alpha > 2$ . Thus, with $\sigma(\zeta) = \sigma_3\zeta^3$ , $\sigma_3 < 0$ , the nonexistence theorem (to be stated below) applies with (2.3) satisfied, $\forall \zeta \in R^1$ , for any $\alpha \in (2,4]$. Note, moreover, that the hypothesis $E(0) \leq 0$ requires, in this case, that

$$\sigma_3 \int_0^1 \int_0^{\tilde{w}_o(x)} \rho^3\,d\rho\,dx \leq -\frac{1}{2}\int_0^2 (\int_0^x \tilde{w}_1(y)dy)^2 dx$$

i.e., that

$$(3.6)\qquad \sigma_3 < 0 \ \ \underline{\text{with}}\ \ |\sigma_3| \geq \frac{2\int_0^1 (\int_0^x \tilde{w}_1(y)dy)^2 dx}{\int_0^1 \tilde{w}_o^4(x)dx}$$

which implies that $\sigma(\zeta) = \sigma_3\zeta^3$ is an admissible nonlinearity, relative to the above nonexistence result only for $\sigma_3$ negative and sufficiently large

in magnitude or for $\sigma_3$ negative and $w(x,0)$ sufficiently large in the sense implied by (3.6). In addition, the initial datum must satisfy

$$(3.7) \qquad \int_0^1 \left(\int_0^x \tilde{w}_o(y)dy\right)\left(\int_0^x \tilde{w}_1(y)dy\right)dx > \left(\frac{\gamma}{2\mu}\right) \int_0^1 \left(\int_0^x \tilde{w}_o(y)dy\right)^2 dx$$

where $\mu = \frac{\alpha-2}{4}$ . With $\sigma(\zeta) = \sigma_3\zeta^3$ , $\sigma_3 < 0$ , $\alpha \in (2,4]$ so that $\mu_{max} = \frac{1}{2}$ : it is sufficient that (3.7) be satisfied with $\mu = \mu_{max}$ We summarize our results for general case in the following.

<u>Theorem I</u>. Consider the initial-boundary value problem (E), (2.1), (2.2). If the initial-data $\tilde{w}_o$, $\tilde{w}_1$ satisfy both (3.7), where $\mu = \frac{\alpha-2}{4}$ , and

$$(3.8) \qquad \int_0^1 \int_0^{\tilde{w}_o(x)} \sigma(\rho)\,d\rho\,dx \le -\frac{1}{2}\int_0^1 \left(\int_0^x \tilde{w}_1(y)dy\right)^2 dx$$

and the nonlinearity $\sigma(\zeta)$ satisfies $\sigma'(0) = 0$ and the growth restriction (2.3), for all $\zeta \in R^1$ , and some $\alpha > 2$ , there can not exist a regular solution of the initial-boundary value problem, i.e., there can not exist a solution $w(x,t) \in C^2([0,1] \times [0,\infty))$ such that $w_x^2(0,\cdot) \in L^\infty[0,\infty) \cap L^1[0,\infty)$ .

Now, suppose that $E(0) < 0$ but $\mathcal{J}(\tilde{w}_o, \tilde{w}_1) \le 0$ . In this case we may first choose $\beta = \beta_o$ such that $\beta_o + 2E(0) = 0$ . Then (3.1), with $\mu = \frac{\alpha-6}{8}$ this time, reduces to

$$(3.9) \qquad \mathcal{F}\mathcal{F}'' - (\mu + 1)\mathcal{F}'^2 \ge -\gamma\mathcal{F}\mathcal{F}' \ , \ 0 \le t < \infty$$

provided $\sigma(\zeta)$ satisfies $\sigma'(0) = 0$ and (2.3) for some $\alpha > 6$ . Inequality (3.9) for $\mathcal{F}(t)$ is formally the same as (3.3) for $\mathcal{F}_o(t)$ and thus the estimate (3.5) applies to $\mathcal{F}(t)^\mu$ . In this case, (3.5), as applied to $\mathcal{F}(t)$, shows that $\mathcal{F}(t)^\mu$ is bounded from below by a function which blows up as $t \to \hat{t}_\infty$ where

$$\hat{t}_\infty = \ln\left\{1 - \left(\frac{\gamma}{\mu}\right)\left[\frac{\mathcal{J}(\tilde{w}_o) + \beta_o t_o^2}{2\beta_o t_o - 2|\mathcal{I}(\tilde{w}_o,\tilde{w}_1)|}\right]\right\}^{-1/\gamma} > 0$$

provided $t_o > \frac{1}{\beta_o}|\mathcal{I}(\tilde{w}_o,\tilde{w}_1)|$ also satisfies

$$\left(\frac{\gamma}{\mu}\right)\frac{\mathcal{J}(\tilde{w}_o) + \beta_o t_o^2}{2\beta_o t_o - 2|\mathcal{I}(\tilde{w}_o,\tilde{w}_1)|} < 1$$

A simple calculation shows that we should choose

$$(3.10) \qquad t_o < \left(\frac{\gamma}{\mu}\right) - \left[\left(\frac{\mu}{\gamma}\right)^2 - \frac{1}{\beta_o}\left(\mathcal{J}(\tilde{w}_o) + 2\left(\frac{\mu}{\gamma}\right)|\mathcal{I}(\tilde{w}_o,\tilde{w}_1)|\right)\right]^{1/2}$$

which, in turn, requires that

$$(3.11) \qquad \frac{\mu}{\gamma} \geq \frac{|\mathcal{I}(\tilde{w}_o,\tilde{w}_1)|}{\beta_o} + \left(\frac{\mathcal{J}(\tilde{w}_o)}{\beta_o} + \frac{|\mathcal{I}(\tilde{w}_o,\tilde{w}_1)|^2}{\beta_o^2}\right)^{1/2}$$

Our results may be summarized as

Theorem II. Consider the initial-boundary value problem (E), (2.1), (2.2). If the initial data satisfy

$$(3.12) \qquad \int_0^1\int_0^{\tilde{w}_o(x)} \sigma(\rho)\,d\rho\,dx < -\frac{1}{2}\int_0^1 \left(\int_0^x \tilde{w}_1(y)dy\right)^2 dx$$

as well as (3.11) with $\mu = \frac{\alpha-6}{8}$ ,

$$(3.13) \qquad \beta_o = -\int_0^1 \left(\int_0^x \tilde{w}_1(y)dy\right)^2 dx + \frac{1}{2}\left|\int_0^1\int_0^{\tilde{w}_o(x)} \sigma(\rho)\,d\rho\,dx\right| > 0$$

where

$$(3.14) \qquad \int_0^1 \left(\int_0^x \tilde{w}_o(y)dy\right)\left(\int_0^x \tilde{w}_1(y)dy\right)dx \equiv \mathcal{I}(\tilde{w}_o,\tilde{w}_1) \leq 0 ,$$

and the nonlinearity $\sigma(\zeta)$ satisfies, $\forall \zeta \in R^1$ , (2.3) for some $\alpha > 6$ and $\sigma'(0) = 0$ , there can not exist a regular solution of the initial-boundary value problem.

Example Consider the problem (E) , (2.1) , (2.2) with $\sigma(\zeta) = \sigma_k \zeta^{2k+1}$ , $k > 0$ not necessarily an integer, $\sigma_k < 0$ , and $\tilde{w}_1(y) = 0$ , $0 \leq y \leq 1$ ;

Clearly $\sigma'(0) = 0$ and (3.12), (3.14) are both satisfied. For $\beta_o$ we take

(3.15) $$\beta_o = \frac{|\sigma_k|}{4(k+1)} \int_0^1 (\tilde{w}_o(x))^{2(k+1)}dx$$

and require of $\tilde{w}_o(x)$ that

(3.16) $$\int_0^1 (\int_0^x \tilde{w}_o(y)dy)^2 dx \leq \left( \frac{\mu^2}{\gamma^2} \right) \frac{|\sigma_k|}{4(k+1)} \int_0^1 (\tilde{w}_o(x))^{2(k+1)}dx$$

so that (3.11) is satisfied. The last condition, i.e., that (2.3) be satisfied, $\forall \zeta \epsilon R^1$, for some $\alpha > 6$, reduces to the condition $6 < \alpha = 2(k+1)$; in other words, (2.3) is satisfied, $\forall \zeta \epsilon R^1$, with $\alpha = 6 + \epsilon, \epsilon > 0$ if $k \geq 2 + \frac{\epsilon}{2}$. For the problem (E), (2.1), (2.2) with $\sigma(\zeta) = \sigma_k < 0$, $k \geq 2 + \frac{\epsilon}{2}$, $\tilde{w}_1(y) = 0$, $0 \leq y \leq 1$, and $\tilde{w}_o(y)$, $0 \leq y \leq 1$ satisfying (3.16), where $\mu = \epsilon/8 > 0$, *a regular solution can not exist.*

Now, suppose that $w(x,t)$ is a regular solution of (E), (2.1), (2.2') with $\sigma(\zeta)$, $\zeta \epsilon R^1$, satisfying (2.3) for some $\alpha > 0$. By lemma 4, $\mathfrak{F}(t)$, as given by (2.10), again satisfies (2.27) and thus the estimate (3.1) holds on $[0,\infty)$, with $\mu = \frac{\alpha-2}{4}$ (so that $\alpha > 2$ is required) in those cases where we may set $\beta = 0$, and with $\mu = \frac{\alpha-6}{8}$ (so that $\alpha > 6$ is required) in those cases where we apply the differential inequality with $\beta \neq 0$. In particular, if conditions (3.2) hold, we may take $\beta = 0$, $\mathfrak{F}(t)$ reduces to $\mathfrak{F}_o(t)$, and $\mathfrak{F}_o(t)$ satisfies (3.3) on $[0,\infty)$. Integration again produces (3.5) and thus the fact that $t_{max} \leq t_\infty < \infty$ provided $\mathcal{H}(\hat{w}_o,\tilde{w}_1) > (\gamma/2\mu)\mathcal{J}(\tilde{w}_o)$. We thus have the following corollary to Theorem I:

<u>Corollary I</u>. Consider the initial-boundary value problem (E), (2.1), (2.2'). If the initial data $\tilde{w}_o, \tilde{w}_1$ satisfy both (3.7), with $\mu = \frac{\alpha-2}{4} > 0$, and (3.8) and $\sigma(\zeta)$ satisfies (2.3), $\forall \zeta \epsilon R^1$, and some $\alpha > 2$, there can not

exist a regular solution of the initial-boundary value probelm, i.e., there can not exist a solution $w(x,t) \in C^2([0,1] \times [0,\infty))$ such that $\sigma'(w(0,t)) < \infty$, $t > 0$.

<u>Remarks</u>. We need not require in Corallary I that $\sigma'(0) = 0$. However, our results certainly do not contraiict the earlier work of Nishida [6], et.al., on initial-value problems (on $R^1$) associated with the system S. For example, if $\sigma(\zeta) = \sigma_1\zeta + \sigma_3\zeta^3$ with $\sigma_1 > 0$, $\sigma_3 > 0$ then (2.3) is satisfied, $\forall \zeta \epsilon R^1$, with $\alpha \geq 4$. Also $\sigma'(\zeta) = \sigma_1 + 3\sigma_3\zeta^2 > 0$, $\forall \zeta \epsilon R^1$ so that S is a hyperbolic system; in this case the work of Nishida [6], et.al., implies the existence of a unique global smooth solution provided $\tilde{w}_o, \tilde{w}_1, \tilde{w}_{o,x}, \tilde{w}_{1,x}$ are sufficiently small in magnitude. For $\sigma(\zeta) = \sigma_1\zeta + \sigma_3\zeta^3$, $\sigma_1 > 0$, $\sigma_3 > 0$, however, (3.8) is never satisfied for any choice of the initial data, no matter how small the data are chosen and thus the nonexistence result of Corollary I does not apply.

The case where $\sigma(\zeta) = \sigma_1\zeta + \sigma_3\zeta^3$ with $\sigma_1 > 0$, $\sigma_3 < 0$ is more delicate. In this case, (2.3) is satisfied $\forall \zeta \epsilon R^1$ if $2 \leq \alpha \leq 4$. Also $\sigma'(0) > 0$, $\sigma'(\zeta) = \sigma_1 - 3|\sigma_3|\zeta^2 > 0$ provided $|\zeta| \leq (\sigma_1/3|\sigma_3|)^{1/2}$. Recall that in establishing the global existence and uniqueness result in [6] it is only required that $\sigma'(0) > 0$; this is because of the crucial observation that if the initial data are chosen sufficiently small, the solution must remain small (for as long as it exists) and thus, it is indeed possible to choose the data so small initially that on the maximal interval of existence of the solution $w(x,t)$, $|w(x,t)| < (\sigma_1/3|\sigma_3|)^{1/2}$. In this case, the

fact that hyperbolicity of S breaks down for values of $\zeta$ such that $|\zeta| > (\sigma_1/3|\sigma_3|)^{1/2}$ has no effect on the fact that globally defined unique smooth solutions exist. With $\sigma(\zeta) = \sigma_1\zeta + \sigma_3\zeta^3$ , $\sigma_1 > 0$ , $\sigma_3 < 0$ , however, the condition (3.8) becomes

$$(3.17) \qquad \int_0^1 \left(\sigma_1 \frac{\tilde{w}_o^2}{2} - \frac{|\sigma_3|}{4}\tilde{w}_o^4\right)dx \le -\frac{1}{2}\int_0^1 \left(\int_0^x \tilde{w}_1(y)dy\right)^2 dx$$

In order to see what this condition implies, vis a vis, the existence theorem of Nishida [6], et.al., consider the simple case in which $\tilde{w}_1(y) \not\equiv 0$ , $0 \le y \le 1$ . and (3.7) is satisfied. Then (3.17) certainly implies the statement that

$\int_0^1 \left(\sigma_1 \frac{\tilde{w}_o^2}{2}(x) - |\sigma_3| \frac{\tilde{w}_o^4}{4}(x)\right)dx < 0$ ; assuming that $\tilde{w}_o(\cdot)$ is at least continuous on $[0,1]$ this will be satisfied if[4]

$$(3.18) \qquad |\tilde{w}_o(x)| \ge \left(\frac{2\sigma_1}{|\sigma_3|}\right)^{1/2} + \delta \quad , \ \delta > 0 \quad .$$

But if (3.18) is satisfied with $\delta$ sufficiently large, i.e., if (3.8) is satisfied with $\sigma(\zeta) = \sigma_1\zeta + \sigma_3\zeta^3$ , $\sigma_1 > 0$ , $\sigma_3 < 0$ , it no longer follows from the results in [6] that $|w(x,t)|$ remains below the bound (namely, $(\sigma_1/3|\sigma_3|)^{1/2}$) which would insure that $\sigma'(w(x,t)) > 0$ , $0 \le x \le 1$ , $t \in [0, t_{max})$. If (3.18) is satisfied then, in general, $\sigma'(0) > 0$ no longer guarantees that $\sigma'(w) > 0$ for as long as smooth solutions exist. Hyperbolicity breaks down and, as noted in Slemrod [9], [10], the global *nonexistence* and breakdown results proven there for data $\tilde{w}_o, \tilde{w}_1$ sufficiently small in magnitude, with $\tilde{w}_{o,x}$ , $\tilde{w}_{1,x}$ large, no longer apply. To summarize: if $\sigma(\zeta) = \sigma_1\zeta + \sigma_3\zeta^3$ , $\sigma_1 > 0$ , $\sigma_3 < 0$ and the data are chosen so that (3.7), (3.8) are satisfied then regular solutions of (E), (2.1), (2.2') can not exist; this result compliments the results of Slemrod [9], [10] concerning

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(4) For $|\tilde{w}_o(x)|$ sufficiently large, therefore, $0 \le x \le 1$ , (3.17) will be satisfied for any data function $\tilde{w}_1(.)$ ; certainly, $|\tilde{w}_o(x)|$, $0 \le x \le 1$ will have to be at least as large as the lower bound (3.18).

global nonexistence of smooth solutions for viscoelastic problems associated with fluids of grade three (it applies to the situation in which the system loses its hyperbolic character during the course of the flow) and does not contradict the global existence results implied by the work of Nishida [6], et. al.

Concerning the initial-boundary value problem (E), (2.1), (2.2') we also have the following direct corollary of Theorem II:

Corollary II. For the initial-boundary value problem (E), (2.1), (2.2') suppose that the initial data $\tilde{w}_o, \tilde{w}_1$ satisfy (3.12), (3.14) and (3.11), with $\mu = \frac{\alpha-6}{8} > 0$ and $\beta_o$ given by (3.13), and the nonlinearity $\sigma(\zeta)$ satisfies, $\forall \zeta \in R^1$, (2.3) for some $\alpha > 6$; then a regular solution of (E), (2.1), (2.2') can not exist.

Remarks. The global nonexistence results contained in the statements of the two theorems above, and the associated Corollaries, carry over in an obvious way to equivalent initial-boundary value problems for (S) of the form

$$(3.19) \quad \begin{cases} w_t - v_x = 0 \\ v_t - \sigma(w)_x + \gamma v = 0 \end{cases} \qquad (0 \leq x \leq 1\ ,\ t > 0)$$
$$w(x,0) = \tilde{w}_o(x)\ ,\ v(x,0) = \tilde{v}_o(x)\ ,\ 0 \leq x \leq 1$$
$$v(0,t) = 0\ ,\ v_x(1,t) = 0\ ,\ t > 0\ .$$

In fact we have already seen that if $w(x,t)$ is a solution of (E), and we define $v(x,t)$ in terms of $w(x,t)$ by (1.21), then the pair $(v,w)$ is a solution of the system (S). If we have initial conditions $w(x,0) = \tilde{w}_o(x)$,

$w_t(x,0) = \tilde{w}_1(x)$ associated with (E) then the corresponding initial conditions associated with the equivalent system (S) are $\tilde{w}_0(x)$ and $\tilde{v}_0(x) = \int_0^x \tilde{w}_1(y)dy$ In the application involving Theorem I, the boundary conditions associated with (E) have the form $w(0,t) = w(1,t) = 0$ and regularity involves the assumption that $w_x^2(0,\cdot) \in L^\infty(0,\infty) \cap L^1[0,\infty)$. In view of (1.21), the associated equivalent boundary conditions on $v(x,t)$ are $v(0,t) = 0$ and $v_x(1,t) = 0$ as $\sigma'(0) = 0$ in this case. In the application involving Corollary I we do not require that $\sigma'(0) = 0$ but we have $w_x(0,t) = 0$ and $w(1,t) = 0$ ; regularity involves the assumption that $\sigma'(w(0,t)) < \infty$, $\forall t > 0$. In this case the associated equivalent boundary conditions on $v(x,t)$ are again $v(0,t) = v_x(1,t) = 0$. We leave to the reader the simple task of carrying over the conclusions of the above global nonexistence results for initial-boundary value problems associated with (E) to equivalent initial-boundary value problems of the form (3.19) which are associated with the system (S).

We now want to turn our attention to the derivation of growth estimates for solutions of initial-boundary value problems associated with (E); these estimates are valid on the maximal time interval $[0, t_{max})$ of a sufficiently smooth solution. While it is possible to derive a variety of such growth estimates from the estimates (2.23), (2.26) which were derived for regular solutions of (E), (2.1), (2.2), and corresponding estimates which can be derived for regular solutions of (E), (2.1), (2.2'), we will confine our attention here to the initial-boundary value problem (E), (2.1), (2.2) and the lower bound on $\mathfrak{F}''$ that is given by (2.23). As we may have $t_{max} = T < \infty$,

the corresponding growth estimate will apply to solutions $w(x,t)$ of (E), (2.1), (2.2) which satisfy $w(x,t) \in C^2([0,1]\times[0,T))$ and $w_x^2(0,\cdot) \in L^\infty[0,T) \cap L^1[0,T)$ .

If we combine (2.23) with the estimate

$$\mathcal{F}'(t)^2 \leq 8 \int_0^1 \left(\int_0^x w(y,t)dy\right)^2 dx \int_0^1 \left(\int_0^x w_t(y,t)dy\right)^2 dx + 8\,\beta(t+t_0)^2$$

use the definition of $\mathcal{F}(t)$ , and then employ the Cauchy-Schwartz inequality, we readily obtain the differential inequality

$$(3.20) \qquad \mathcal{F}\mathcal{F}'' - \frac{(2+\alpha-\gamma)}{8}\mathcal{F}'^2 \geq -(\gamma+1)\mathcal{F}^2 - \alpha\,\mathcal{F}(\beta + \varkappa_T\sigma'^2(0) + 2E(0))$$

for $0 \leq t \leq t_{max} = T$ , where

$$(3.21) \qquad \varkappa_T = \frac{2}{\alpha}\|w_x^2(0,\cdot)\|_{L^\infty(0,T)} + \frac{1}{2\gamma}\|w_x^2(0,\cdot)\|_{L^1(0,T)}$$

If $t_{max} = \infty$ , and we are dealing with a regular solution of (E), (2.1), (2.2), then $\varkappa_T$ in (3.20) must be replaced by $\varkappa_\infty$ and (3.20) holds for $0 \leq t < \infty$ . We first consider those cases in which

$$(3.22) \qquad \varkappa_T\sigma'^2(0) + 2E(0) \leq 0$$

($\varkappa_\infty\sigma'^2(0) + 2E(0) \leq 0$ , if $t_{max} = \infty$ ) . If $\sigma'(0) = 0$ then (3.22) reduces to the requirement that $E(0)$ be nonpositive; otherwise we require that the initial energy satisfy $E(0) \leq -\frac{1}{2}\varkappa_T\,\sigma'^2(0)$ . If (3.22) holds then we may take $\beta = 0$ in (3.20) and reduce the differential inequality to

$$(3.23) \qquad \mathcal{F}_0\mathcal{F}_0'' - \frac{(2+\alpha-\gamma)}{4}\mathcal{F}_0'^2 \geq -(\gamma+1)\mathcal{F}_0^2 \quad , \quad 0 \leq t \leq t_{max}$$

We now set $\mu = (\alpha-\gamma-2)/4$ and require that (2.3) hold, $\forall \zeta \in R^1$ , for some

$\alpha > \gamma + 2$ ; then $\mu > 0$ and (3.23) becomes

$$(3.24) \qquad \mathcal{F}_o\mathcal{F}_o'' - (\mu+1)\mathcal{F}_o'^2 \geq -(\gamma+1)\mathcal{F}_o^2 \quad , \; 0 \leq t \leq t_{max}$$

We note that differential inequalities of the form (3.24) have appeared previously in the literature (e.g. [16], §II) . Following the analysis in [16] we set $\mathcal{L}(t) = \mathcal{F}_o^{-\mu}(t)$ and note that $\mathcal{L}'(0) = -\mu\mathcal{F}_o^{-(\mu+1)}(0)\mathcal{F}_o'(0) < 0$ if $\mathcal{F}_o'(0) > 0$ , i.e., if $\mathcal{J}(\tilde{w}_o,\tilde{w}_1) > 0$ ; under these circumstances, we have $\mathcal{L}'(t) < 0$ on some interval $[0,\tau)$ . By (3.24) then $\mathcal{L}''(t) \leq \mu(\gamma+1)\mathcal{L}(t)$ and thus, for $t \in [0,\eta]$ we may multiply on both sides by $\mathcal{L}'(t)$ and integrate so as to obtain

$$(3.25) \qquad \mathcal{L}'(t)^2 - \mathcal{L}'(0)^2 \geq \mu(\gamma+1)(\mathcal{L}(t)^2 - \mathcal{L}(0)^2)$$

Clearly the estimate (3.25) may br rewritten in the form

$$(3.26) \qquad (\mathcal{L}'(t) + \sqrt{\mu(\gamma+1)}\,\mathcal{L}(t))(\mathcal{L}'(t) - \sqrt{\mu(\gamma+1)}\,\mathcal{L}(t))$$
$$\geq (\mathcal{L}'(0) + \sqrt{\mu(\gamma+1)}\,\mathcal{L}(0))\;(\mathcal{L}'(0) - \sqrt{\mu(\gamma+1)}\,\mathcal{L}(0))$$

If $\mathcal{L}'(0) < -\sqrt{\mu(\gamma+1)}\,\mathcal{L}(0)$ , then by the assumed smoothness of $w(x,t)$ , for $0 \leq t \leq t_{max}$ , it follows that neither factor on the left-hand side of (3.26) can change sign on $[0, t_{max}]$. Therefore, for $0 \leq t \leq t_{max}$ , $\mathcal{L}'(t) < -\sqrt{\mu(\gamma+1)}\,\mathcal{L}(t)$ , which implies that $\mathcal{L}(t)\exp(\sqrt{\mu(\gamma+1)}\,t) \leq \mathcal{L}(0)$ , or

$$\mathcal{F}_o(t) \geq \mathcal{F}_o(0)\exp\left(\sqrt{\frac{\gamma+1}{\mu}}\,t\right) \; , \; 0 \leq t \leq t_{max}$$

Now, the condition that $\mathcal{L}'(0) < -\sqrt{\mu(\gamma+1)}\,\mathcal{L}(0)$ is equivalent to the requirement that

$$\mathcal{J}(\tilde{w}_o,\tilde{w}_1) \geq \frac{1}{2}\sqrt{\frac{\gamma+1}{\mu}}\;\mathcal{G}(\tilde{w}_o) \equiv \sqrt{\frac{\gamma+1}{\alpha-\gamma-2}}\;\mathcal{G}(\tilde{w}_o)$$

where $\alpha > \gamma+2$ . We may, therefore, state the following

Theorem III. Let $w(x,t) \in C^2([0,1] \times [0,T))$ be any solution of (E), (2.1), (2.2), $T \leq \infty$ , for which $\varkappa_T < \infty$ and assume that the nonlinearity $\sigma(\zeta)$ satisfies (2.3) , $\forall \zeta \in R^1$ , for some $\alpha > \gamma+2$ . Then, if the initial data $\tilde{w}_o$ , $\tilde{w}_1$ satisfy

(i) $E(0) \leq -\frac{1}{2} \varkappa_T \sigma'(0)^2$

(ii) $\mathcal{J}(\tilde{w}_o,\tilde{w}_1) \geq \left(\frac{\gamma+1}{\alpha-\gamma-2}\right)^{1/2} \mathcal{I}(\tilde{w}_o)$

it follows that

(3.27) $$\int_o^1 \left(\int_o^x w(y,t)dy\right)^2 dx \geq \mathcal{I}(\tilde{w}_o) \exp\left(2\sqrt{\frac{\gamma+1}{\alpha-\gamma-2}}\, t\right) \quad , 0 \leq t < T$$

Remarks. If condition (ii) of the theorem is not satisfied, i.e., if $\mathfrak{F}_o'(0) \leq \frac{1}{2}\sqrt{\frac{\gamma+1}{\mu}}\ \mathfrak{F}_o(0)$ with $\mu = (\alpha-\gamma-2)/4$ and

$$\varkappa_T \sigma'^2(0) + 2E(0) < 0$$

then we would work instead with the differential inequality (3.20). We would first choose $\beta = \beta_o$ such that $\beta_o + \varkappa_T\sigma'^2(0) + 2E(0) = 0$ and then choose $t_o$ so large that

$$\mathcal{J}(\tilde{w}_o,\tilde{w}_1) + \beta_o t_o \geq \frac{1}{2}\sqrt{\frac{\gamma+1}{\mu}}\ \mathcal{I}(\tilde{w}_o)$$

where, in view of (3.20) we now have $\mu = \frac{\alpha-\gamma-6}{8}$ and $\sigma(\zeta)$ is required to satisfy (2.3), $\forall \zeta \in R^1$ , and some $\alpha > \gamma+6$ ; an increasing exponential lower bound for $\mathcal{I}(w(x,t))$ of the form (3.27) again follows. If $\sigma'(0) = 0$ the condition (ii) of the Theorem reduces to the requirement that (3.12) be satisfied. Finally, a series of simple estimates, employing only the

Schwartz inequality, readily establishes that (3.27) implies an exponentially increasing lower bound for $\|w(\cdot,t)\|^2_{L^2(0,1)}$ on $[0, t_{max}]$ .

Many of the results of this section may be applied to initial-boundary value problems associated with the model of nonlinear fluid viscoelastic response considered in [9], [10]. While our results to not apply directly to the problem of shearing perturbations from a steady rectilinear flow with associated no-slip boundary conditions, they do apply to the following situation: the evolution equation in the Slemrod model in [9], [10] is given by (1.13), i.e.,

(1.13) $\rho \mathcal{V}_t(x,t) = \sigma \left(\int_0^\infty e^{-\gamma s}\mathcal{V}_x(x,t-s)ds\right)_x$

and there is a prescribed, associated smooth velocity history given by (1.15c), i.e.,

(1.15c) $\mathcal{V}(x,\tau) = \mathcal{V}_0(x,\tau)$ , $-\infty < \tau \leq 0$

Suppose we associate with (1.13) the homogeneous boundary conditions

(3.28) $\mathcal{V}_x(0,t) = 0$ , $\mathcal{V}_x(1,t) = 0$ , $t > 0$

and, following the analysis in [9], [10] define

(3.29a) $v(x,t) = \int_0^\infty e^{-\gamma s}\mathcal{V}_t(x,t-s)ds$ , $0 \leq x \leq 1$ , $t > 0$

(3.29b) $w(x,t) = \int_0^\infty e^{-\gamma s}\mathcal{V}_x(x,t-s)ds$ , $0 \leq x \leq 1$ , $t > 0$

It then follows that $(v,w)$ satisfy (S) and, thus, $w(x,t)$ , as given by (3.29b) , satisfies (E) . Also, in view of (3.28), we have $w(0,t) = w(1,t) = 0$ , and by (1.15c), (3.29b) we have

(3.30) $\quad \tilde{w}_o(x) = \int_{-\infty}^{0} e^{\gamma s} \frac{\partial}{\partial x} \mathcal{V}_o(x,s)ds \;,\; \tilde{w}_1(x) = \int_{-\infty}^{0} e^{\gamma s} \frac{\partial^2}{\partial x \partial s} \mathcal{V}_o(x,s)ds$

By Theorem I , therefore if the nonlinearity $\sigma(\zeta)$ in (1.13) satisfies $\sigma'(0) = 0$ and the growth restriction (2.3), $\forall \zeta \in R^1$ , and some $\alpha > 2$ , and the initial velocity history $\mathcal{V}_o(x,\tau)$ , $-\infty < \tau \leq 0$ , satisfies the conditions implied by (3.7), (3.8), where $\tilde{w}_o$ , $\tilde{w}_1$ are given by (3.30), it follows that there can not exist a solution $\mathcal{V}(x,t)$ of the initial-history boundary value problem (1.13), (1.15c), (3.28) which is such that $w(x,t)$ , as defined by (3.29b) is regular, i.e., satisfies $w(x,t) \in C^2([0,1] \times [0,\infty))$ and $w_x^2(0,\cdot) \in L^\infty(0,\infty) \cap L^1[0,\infty)$. However, by the simple relations

$$\mathcal{V}_t = \sigma(w)_x$$
$$\mathcal{V}_x = v_x + \gamma w$$

which are a direct consequence of (3.29a,b) it then follows that there can not exist a solution $\mathcal{V}(x,t)$ of the initial-history boundary value problem (1.13), (1.15c), (3.28) which satisfies $\mathcal{V}(x,t) \in C^2([0,1] \times [0,\infty))$ with $\mathcal{V}_{xx}^2(0,\cdot) \in L^\infty[0,\infty)$ . In one sense this result extends the work of Slemrod [9], [10] to the situation where $\sigma(\zeta)$ satisfies $\sigma'(0) = 0$ , a situation that can not possibly be handeled by the Riemann invariant argument approach in [9], [10]. On the other hand the result is weaker than the type of results contained in [9], [10] in as much as the Riemann invariant argument employed there (for $\sigma'(0) > 0$) yields global nonexistence of a solution in $C^1([0,1] \times [0,\infty))$ and shows that such global nonexistence occurs as a results of finite - time blow up of the space-time gradient $(\mathcal{V}_x, \mathcal{V}_t)$ of solutions. We leave for the reader the simple task of carrying over the other global nonexistence results and growth estimates obtained in this

section, for initial-boundary value problems associated with (E), to the corresponding initial-history boundary value problems governing nonlinear viscoelastic response; the results of Corollary II, in particular, may be carried over with (3.28) replaced by the homogeneous boundary data

$$\mathcal{V}_{xx}(0,t) = \mathcal{V}_x(1,t) = 0 \ , \ t > 0 \ .$$

NOTE: This work was preformed while the author was visiting the School of Mathematics at the University of Minnesota.

## References

1. Zabusky, N.J., "Exact Solution for Vibrations of a Nonlinear Continuous Model String", Journ. Math. Phys., 3, (1962), 1028-1039.

2. Lax, P.D., "Development of Singularities of Solutions of Nonlinear Hyperbolic Partical Differential Equations", Journ. Math. Phys., 5, (1964), 611-613.

3. MacCamy, R.C. and V.J. Mizel, "Existence and Non-existence in the Large of Solutions of Quasilinear Wave Equations", Arch. Rat. Mech. Anal., 25, (1967), 299-320.

4. MacCamy, R.C., "Existence, Uniqueness, and Stability of Solutions of the Equation $U_{tt} = \frac{\partial}{\partial x}(\sigma(U_x) + \lambda(U_x)U_{xt})$", Ind. Univ. Math. Journ., 20, (1970), 231-237.

5. Greenberg, J.M., R.C. MacCamy, and V.J. Mizel, "On the Existence, Uniqueness, and Stability of Solutions of the Equation $\sigma'(U_x)U_{xx} + \lambda U_{xtx} = \rho_o U_{tt}$" Journ. Math. Mech., 17, (1968), 707-728.

6. Nishida, T., Nonlinear Hyperbolic Equations and Related Topics in Fluid Dynamics, Publications Mathematiques D'Osay 78.02, Universite de Paris - Sud, Department de Mathématique, (1978).

7. Matsumura, A., "Global Existence and Asymptotics of the Solutions of the Second-Order Quasilinear Hyperbolic Equations with First Order Dissipation", Publications Research Institute Mathematical Sciences, Kyoto University, Series A 13 (1977), 349-379.

8. Courant, R., Friedrichs, K.O., and H. Lewy, "Uber die partiellen Differential gleichungen der Mathematischen Physik", Math. Ann., 100, (1928), 32-74.

9. Slemrod, M., "Instability of Steady Shearing Flows in a Nonlinear Viscoelastic Fluid", *Arch. Rat. Mech. Anal.*, *68*, (1978), 211-225.

10. Slemrod, M., "Damped Conservation Laws in Continuum Mechanics", in *Nonlinear Analysis and Mechanics*, *vol. III*, Pitman Pub., (1978), 135-173.

11. MacCamy, R.C., "A Model for One-Dimensional Nonlinear Viscoelasticity", *Q. Appl. Math.*, *35*, (1977), 21-33.

12. Dafermos, C.M. and J.A. Nohel, Energy Methods for Nonlinear Hyperbolic Volterra Integrodifferential Equations", *Comm. P.D.E.*, *4*, (1979), 219-278.

13. Bloom, F., "A Lower Bound for the Norm of the Solution of a Nonlinear Volterra Equation in One-Dimensional Viscoelasticity", submitted.

14. Nohel, J.A., "A Forced Quasilinear Wave Equation with Dissipation", *Proc. of EQUADIFF IV*, *Lecture Notes in Mathematics*, 703, (1977), 318-327.

15. MacCamy, R.C., "An Integrodifferential Equation with Applications in Heat Flow", *Q. Appl. Math.*, *35*, (1977), 1-19.

16. Levine, H.A., "Instability and Nonexistence of Global Solutions to Nonlinear Wave Equations of the Form $Pu_{tt} = -Au + \mathcal{F}(u)$," *Trans. Amer. Math. Soc.*, *190*, (1974), 1-21.

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